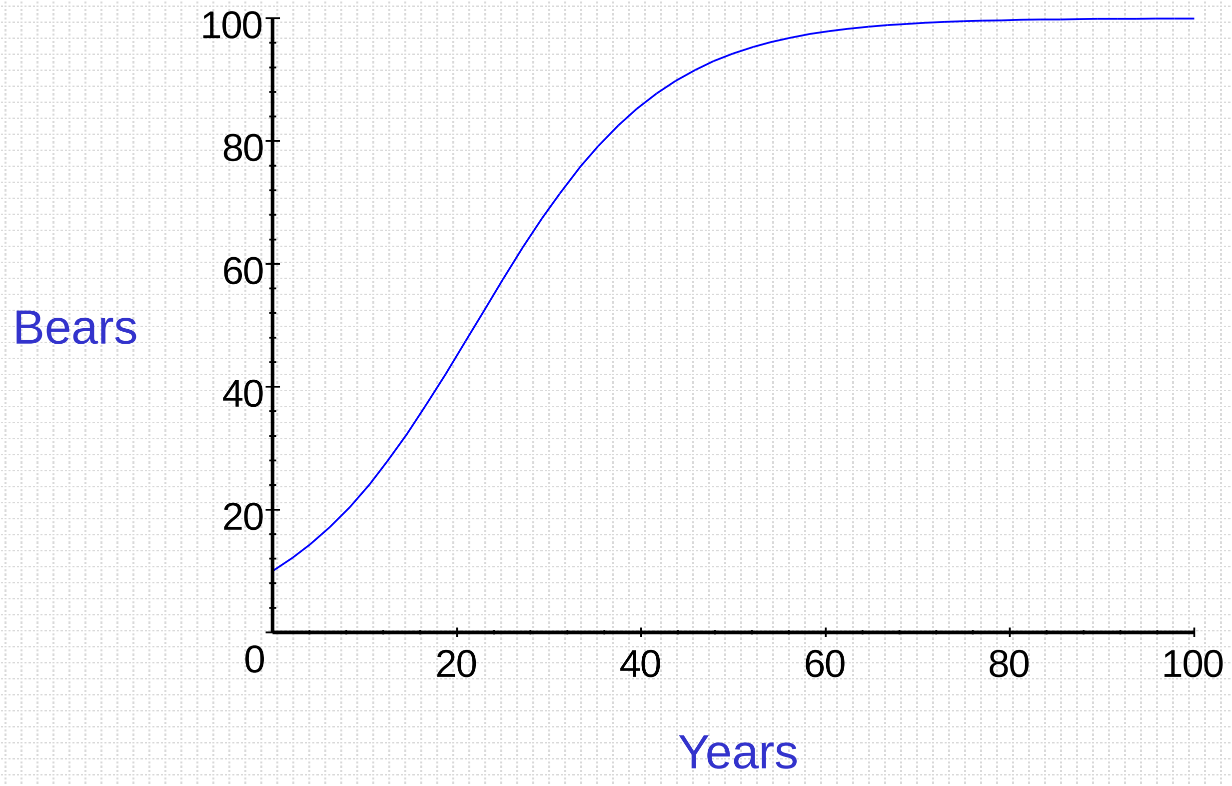
6.5: Logistic Growth Model



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We have used the exponential growth equation $y = y_0 e^{kt}$ to represent population growth.

The exponential growth equation occurs when the rate of growth is proportional to the amount present.

If we use *P* to represent the population, the differential equation becomes:

$$\frac{dP}{dt} = kP$$

The constant *k* is called the *relative growth rate*.

$$\frac{dP/dt}{P} = k$$

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The population growth model becomes: $P = P_0 e^{kt}$

However, real-life populations do not increase forever. There is some limiting factor such as food, living space or waste disposal.

There is a maximum population, or carrying capacity, *M*.

A more realistic model is the *logistic growth model* where growth rate is proportional to both the amount present (*P*) and the fraction of the carrying capacity $\left(\frac{M-P}{M}\right)$ that remains:

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The equation then becomes:

$$\frac{dP}{dt} = kP\left(\frac{M-P}{M}\right)$$

The equation is normally written this way:

Logistics Growth Model

$$\frac{dP}{dt} = \frac{k}{M}P\left(M-P\right)$$

We can solve this differential equation to find the general form of the solution curve.

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Logistics Differential Equation

$$\frac{dP}{dt} = \frac{k}{M} P\left(M - P\right)$$

$$\frac{1}{P\left(M - P\right)} dP = \frac{k}{M} dt$$

$$\frac{1}{\cancel{M}}\left(\frac{1}{P} + \frac{1}{M - P}\right) dP = \frac{k}{\cancel{M}} dt$$

$$\ln P - \ln\left(M - P\right) = kt + C$$

$$\ln \frac{P}{M - P} = kt + C$$

$$\frac{1}{P\left(M - P\right)} = \frac{A}{P} + \frac{B}{M - P}$$

$$1 = A\left(M - P\right) + BP$$

Partial Fractions

$$1 = AM - AP + BP$$

$$1 = AM \qquad 0 = -AP + BP$$

$$AP = BP$$

$$\frac{1}{M} = A \qquad A = B$$

$$\frac{1}{M} = B$$

$\rightarrow$

Logistics Differential Equation

$$\frac{dP}{dt} = \frac{k}{M} P(M-P)$$

$$\frac{1}{P(M-P)} dP = \frac{k}{M} dt$$

$$\cancel{\frac{1}{M}}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \cancel{\frac{k}{M}} dt$$

$$\ln P - \ln(M-P) = kt + C$$

$$\ln \frac{P}{M-P} = kt + C$$

$$\frac{P}{M-P} = e^{kt+C}$$

$$\frac{M-P}{P} = e^{-kt-C}$$

$$\frac{M}{P} - 1 = e^{-kt-C}$$

$$\frac{M}{P} = 1 + e^{-kt-C}$$

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Logistics Differential Equation

$$\frac{P}{M-P}=e^{kt+C}$$

$$\frac{M-P}{P}=e^{-kt-C}$$

$$\frac{M}{P}-1=e^{-kt-C}$$

$$\frac{M}{P}=1+e^{-kt-C}$$

$$P=\frac{M}{1+e^{-kt-C}}$$

$$P=\frac{M}{1+e^{-C}\cdot e^{-kt}}$$

$$Let\quad A=e^{-C}$$

$$\boxed{P=\frac{M}{1+Ae^{-kt}}}$$

$\rightarrow$

Logistics Growth Model

$$\frac{dP}{dt} = \frac{k}{M} P\left(M - P\right)$$

Solution Curve to Logistic Growth Model

$$P = \frac{M}{1 + Ae^{-kt}}$$

$\rightarrow$

A park can support no more than 200 deer. There are 30 deer in the park now. Assume a logistic growth model and *k* = 0.15.

a. Find the logistic growth model.

$$\frac{dP}{dt} = \frac{k}{M} P(M - P)$$

$$\boxed{\frac{dP}{dt} = \frac{.15}{200} P(200 - P)}$$

A park can support no more than 200 deer. There are 30 deer in the park now. Assume a logistic growth model and k = 0.15. $P_0 = 30$

b. Find an equation for the population at time t.

$$P = \frac{M}{1 + Ae^{-kt}}$$

$$P = \frac{200}{1 + Ae^{-.15t}}$$

$$30 = \frac{200}{1 + A\cancel{e^{-.15(0)}}} \quad 1$$

$$30 = \frac{200}{1 + A}$$

$$1 + A = \frac{200}{30}$$

$$A = \frac{200}{30} - 1 \approx 5.6667$$

$$\boxed{P = \frac{200}{1 + 5.6667e^{-.15t}}}$$

A park can support no more than 200 deer. There are 30 deer in the park now. Assume a logistic growth model and k = 0.15.

c. Draw a slope field with the solution curve.

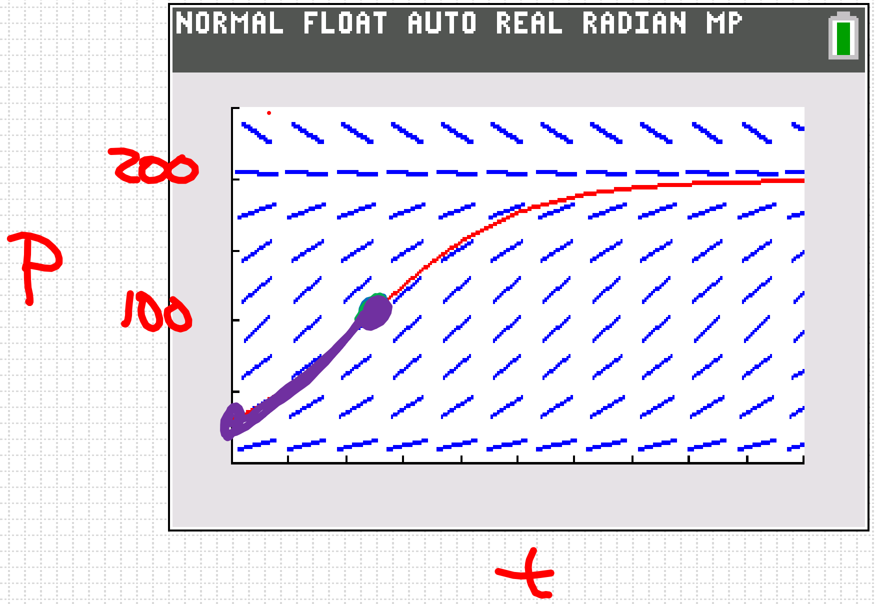
d. For what value of *P* is the population growing the fastest? $\frac{M}{2} \rightarrow 100$

e. What is the limit of *P* as *t* approaches infinity? $\lim_{t \to \infty} P(t) = M = 200$

f. When is *P* increasing at an increasing rate? Decreasing rate?

$30 < P < 100$

$100 < P < 200$



Example:

Logistic Growth Model

Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears.

Assuming a logistic growth model, when will the bear population reach 75?

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$$P = \frac{M}{1 + Ae^{-kt}} \qquad M = 100 \qquad P_0 = 10 \qquad P_{10} = 23$$

$$P = \frac{100}{1 + Ae^{-kt}}$$

$$10 = \frac{100}{1 + A\cancel{e^{-k(0)}}\ 1}$$

$$10 = \frac{100}{1 + A}$$

$$A = 9$$

$$P = \frac{100}{1 + 9e^{-kt}}$$

$$23 = \frac{100}{1 + 9e^{-k(10)}}$$

$$1 + 9e^{-10k} = \frac{100}{23}$$

$$9e^{-10k} = \frac{100}{23} - 1$$

$$\ln e^{-10k} = \ln \frac{\frac{100}{23} - 1}{9}$$

$$-10k = \ln\left(\frac{\frac{100}{23} - 1}{9}\right)$$

$$k = \frac{\ln\left(\frac{\frac{100}{23} - 1}{9}\right)}{-10}$$

$$k \approx .09889\ldots$$

→

$$P = \frac{M}{1 + Ae^{-kt}}$$

$M = 100$ $P_0 = 10$ $P_{10} = 23$ $P_{75} = ??$

$$P = \frac{100}{1 + 9e^{-.09889t}}$$

$$75 = \frac{100}{1 + 9e^{-kt}}$$

$$1 + 9e^{-kt} = \frac{100}{75}$$

$$e^{-kt} = \frac{\frac{100}{75} - 1}{9}$$

$$t = \frac{\ln\left(\frac{\frac{100}{75} - 1}{9}\right)}{-k} \approx 33.328 \text{ YRS}$$

FROM $t = 0$.

$\therefore$ 23.328 YRS FROM NOW.

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Homework:

Section 6.5 – Logistic Growth FDWK